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Nonlinear electromagnetics in chiral media: Self-action of waves

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We report the development of a theory of the self-action of waves in nonlinear chiral media. The basic equations of nonlinear electromagnetism in a chiral medium are reduced to a set of nonlinear coupled Schrödinger equations (NCSE). A partial solution of the NCSE in the form of planar waves and their stability with respect to small perturbations are examined. The Hamiltonian form of the NCSE, as well as conservation principles and the soliton solutions of the NCSE are presented. The presence of chirality is shown to result in an asymmetry of the solitonic spectrum with respect to the handedness of the field. A theory of the interaction of dark and bright solitons in defocusing chiral media is developed. The obtained results, their possible generalization, and their applications are discussed.

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I. INTRODUCTION

Chiral media are isotropic birefringent substances whose microstructure is handed. The handedness imparts them special electromagnetic properties that were first recognized at optical frequencies almost 175 years ago [1]. When an elliptically polarized plane wave is incident on a slab made of a linear chiral material, the vibration ellipse of the transmitted plane wave is found to be different from that of the incident plane wave in two respects: (i) the tilt angle and (ii) the axial ratio [2]. Following Pasteur [1], chiral media are sometimes called *natural optically active media.*

There is currently considerable interest in studying chiral media, owing to their extraordinary electromagnetic and optical properties and a variety of actual and potential applications in radiophysics, microwaves, and visible light optics [2–4]. This interest has stimulated the development of the linear electrodynamics of chiral media. Theoretical investigations, normally carried out in the frequency domain, exploit the Bohren transform to convert the electric and the magnetic field phasors into frequency-domain Beltrami fields [2]. The success of the Bohren transform for frequency-domain fields in a chiral medium has led recently to the development of the Beltrami-Maxwell postulates for time-dependent Beltrami fields in any material medium [5]. The electrodynamic consequences of this formalism have been examined in free space [6,7], as well as in linear homogeneous media [8].

Attention must now shift to theoretical analysis of nonlinear phenomena in chiral media, because recent experimental results are intriguing and promise dividends in biophysics [9] and materials sciences [10]. Recently, the application of chiral synthetic materials to the operation of distributed feedback lasers has been pointed out by Flood and Jaggard [11]. In our opinion, in order to develop the theoretical basis of electrodynamics of nonlinear chiral media the following problems should be analyzed: (i) the self-action of waves, (ii) the interaction of n waves with different frequencies, and (iii) wave kinetics.

Our aim in this paper is to consider the first of the three problems; that is, to develop a theory of the self-action of waves in nonlinear chiral media. The main feature of a chiral medium is its circular birefringence, which is strongly frequency dependent [12]. Thus, chiral media are akin to other media with strong wave-wave interactions at a given frequency, such as nonlinear birefringent crystals [13,14], bimodal nonlinear optical fibers [15–19], and plasmas [20].

An important feature of a chiral medium is that its cubic nonlinearity is tensorial [21]. Together with birefringence, this property entails that the self-action in a nonlinear chiral medium must be described by a general system of two nonlinear coupled Schrödinger equations (NCSE).

Note that only specific representations of the NCSE are conventionally used for analytical treatment. When the medium is unirefringent and the nonlinearity can be described by a scalar parameter, the NCSE reduce to the exactly integrable Manakov system [22] that can be solved using the inverse scattering method. The Manakov system describes the effect of vector self-focusing or self-defocusing in Kerr dielectrics. In general, the NCSE are not integrable [23,24] and either approximate analytical or numerical methods must be applied. Haelterman and Sheppard [25–27] considered a unirefringent medium with tensorial nonlinearity. The NCSE for opposing waves with the same phase velocities were examined by Malomed and Tasgal [18]. Eleonskii *et al.* [16] and Bhakta [20] analyzed the general representation of the NCSE. The Karpman-Maslov [28] and Hirota [20] methods are potentially useful for numerical integration of the NCSE.

Here, we develop a phenomenological theory of the self-action of waves in nonlinear chiral media. On no account can we therefore ignore birefringence. Our theory is based on general methods of the phenomenological nonlinear optics [21,29] extended to chiral media and on the Beltrami-Maxwell formalism adapted to nonlinear problems.

This paper is arranged as follows: In Sec. II we give the basic equations of nonlinear electromagnetism in a chiral medium and reduce them to the NCSE form. A partial solution of the NCSE in the form of planar waves and the problem of their stability with respect to small perturbations are considered in Sec. III. In Sec. IV we present the Hamiltonian form of the NCSE as well as conservation principles. Soliton solutions of the NCSE are discussed in Sec. V, particularly a theory of the interaction of dark and bright solitons in defocusing chiral media. The obtained results, their possible generalization, and their application are discussed in Sec. VI.

II. BASIC RELATIONS

We start from the Maxwell curl postulates expressed in matrix notation as

$$\nabla\times\tilde{\mathbf{U}}=\partial_t\tilde{\mathbf{V}} \tag{1}$$

with the six-vectors $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ given by

$$\tilde{\mathbf{U}}=\begin{bmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{bmatrix},\quad \tilde{\mathbf{V}}=\begin{bmatrix}-\tilde{\mathbf{B}}\\ \tilde{\mathbf{D}}\end{bmatrix}.$$

Here and hereafter in all six-dimensional equations it is assumed that

$$\nabla\equiv\begin{bmatrix}\nabla & 0\\ 0 & \nabla\end{bmatrix}.$$

The constitutive relations of a nonlinear chiral medium can be written in matrix form as

$$\tilde{\mathbf{V}}=\hat{\Lambda}\tilde{\mathbf{U}}+\tilde{\mathbf{V}}^{\mathrm{NL}}. \tag{2}$$

The first term on the right-hand side of this equation corresponds to the linear approximation, with [30]

$$\hat{\Lambda}\tilde{\mathbf{U}}=\int_{-\infty}^{t}\Lambda(t-\tau)\tilde{\mathbf{U}}(\tau)d\tau . \tag{3}$$

The constitutive operator $\Lambda(t)$ has the form

$$\Lambda(t)=\begin{bmatrix}-\gamma(t)I & -\mu(t)I\\ \epsilon(t)I & -\gamma(t)I\end{bmatrix}, \tag{4}$$

where I is the 3×3 unit matrix, while the scalars $\epsilon(t)$ and $\mu(t)$, as well as the pseudoscalar $\gamma(t)$, are null-valued for negative values of t. We observe here that the most general, isotropic, linear, homogeneous medium can be characterized by the chosen $\Lambda(t)$ [31,32]. In particular, $\gamma(t)$ delineates the linear part of the chiral property of the medium, it being identically null for an achiral isotropic medium. Typically, $\gamma(t)$ represents a small first-order—but a highly significant and easily observable [33,34]—effect.

The second term on the right-hand side of (2) describes the contribution from nonlinearity, which is assumed to be cubic and small compared to the first one. Parenthetically, although quadratic nonlinearity is possible, we consider only cubic nonlinearity because our aim, as noted above, is to analyze the self-action of waves. Thus, the nonlinear term in (2) can be represented by

$$\tilde{\mathbf{V}}^{\mathrm{NL}}=\int_0^\infty\int_0^\infty\int_0^\infty\Delta\Lambda_{ij}(\tau_1,\tau_2,\tau_3)\tilde{U}_i(t-\tau_1-\tau_2)\times\tilde{U}_j(t-\tau_1-\tau_2-\tau_3)\times\tilde{\mathbf{U}}(t-\tau_1)d\tau_1d\tau_2d\tau_3 , \tag{5}$$

where $\Delta\Lambda_{ij}(\tau_1,\tau_2,\tau_3)$ is the nonlinear constitutive parameters tensor (NCPT) of the chiral medium.

A. Equations for a beam

Let us consider a monochromatic beam $\tilde{\mathbf{U}}(\mathbf{R},t)=\mathbf{U}(\mathbf{R})e^{-j\omega t}$, with its amplitude $\mathbf{U}(\mathbf{R})$ independent on t. Note that only the real part of $\tilde{\mathbf{U}}(\mathbf{R},t)$ is observable. Here $\mathbf{R}=(x,\mathbf{r})$ in the three-dimensional position vector, while $\mathbf{r}=(y,z)$ is the two-dimensional position vector in the YZ plane.

From (3), we now obtain $\hat{\Lambda}\tilde{\mathbf{U}}=\Lambda(\omega)\mathbf{U}$, where

$$\Lambda(\omega)=\begin{bmatrix}-j\gamma(\omega)I & -\mu(\omega)I\\ \epsilon(\omega)I & -j\gamma(\omega)I\end{bmatrix}, \tag{6}$$

while $\epsilon(\omega)$, $\mu(\omega)$, and $j\gamma(\omega)$ are the Fourier transforms of $\epsilon(t)$, $\mu(t)$, and $\gamma(t)$. We shall use $\Lambda(\omega)$ exclusively hereafter and denote it simply by Λ. From (1) we find that

$$\nabla\times\mathbf{U}=-j\omega\Lambda\mathbf{U}-j\omega\mathbf{V}^{\mathrm{NL}} , \tag{7}$$

where $\mathbf{V}^{\mathrm{NL}}=\Delta\Lambda_{ij}(\omega)U_iU_j^*\mathbf{U}$, and $\Delta\Lambda_{ij}(\omega)$ represents the NCPT in the frequency domain [35].

Next, let us make use of the conventional definition of the Beltrami fields [5] as $\mathbf{Q}_\pm=\mathbf{E}\pm j\eta\mathbf{H}$ in a chiral medium, where the nonlinear impedance of the medium is defined by $\eta=\sqrt{(\mu+\Delta\mu)/(\epsilon+\Delta\epsilon)}$, and $\Delta\mu$ and $\Delta\epsilon$ are the nonlinear corrections to μ and ϵ, respectively. Only the nonlinear terms couple the Beltrami fields $\mathbf{Q}_\pm$; consequently, the Maxwell curl postulates for them take a very simple form:

$$\nabla\times\mathbf{Q}_\pm=\pm K_{1,2}\mathbf{Q}_\pm\pm f_{1,2}\mathbf{Q}_\pm , \tag{8}$$

wherein $K_{1,2}=\omega(\sqrt{\epsilon\mu}\mp\gamma)$. We emphasize that $\mathbf{Q}_\pm$ are not linear combinations of $\mathbf{E}$ and $\mathbf{H}$, because η depends on $\mathbf{E}$ and $\mathbf{H}$. Finally, the functions $f_{1,2}$ contain the nonlinear behavior of the medium and, according to (5), may be generally represented as

$$f_{1,2}=\alpha_{1,2}|\mathbf{Q}_+|^2+\beta_{1,2}|\mathbf{Q}_-|^2+2\delta_{1,2}\mathrm{Re}(\mathbf{Q}_+\cdot\mathbf{Q}_-^*) , \tag{9}$$

where $\alpha_{1,2}$, $\beta_{1,2}$, and $\delta_{1,2}$ are scalar nonlinear coefficients [36]. Here and hereafter, the asterisk stands for the complex conjugate.

Let us seek a solution to (8) by using the factorizations

$$\begin{aligned} \mathbf{Q}_+(\mathbf{R}) &= [\mathbf{e}^+ u(\mathbf{r},x) + \mathbf{e}_x u_l(\mathbf{r},x)] e^{jK_1 x} , \\ \mathbf{Q}_-(\mathbf{R}) &= [\mathbf{e}^- v(\mathbf{r},x) + \mathbf{e}_x v_l(\mathbf{r},x)] e^{jK_2 x} , \end{aligned} \tag{10}$$

where the unit vectors $\mathbf{e}^{\pm} = \{0, 1, \pm j\}/\sqrt{2}$, $\mathbf{e}_x = \{1,0,0\}$, while $u(\mathbf{r},x)$, $v(\mathbf{r},x)$, $u_l(\mathbf{r},x)$, and $v_l(\mathbf{r},x)$ are slowly varying functions of x and rapidly varying ones with respect to $\mathbf{r}$. Longitudinal components have been included in (10) because the beam is not rigorously transverse. As shown in Appendix A, approximate equations for the transverse fields $u(\mathbf{r},x)$ and $v(\mathbf{r},x)$ may be found to be

$$\begin{aligned} jK_1 \partial_x u &= -\tfrac{1}{2}\nabla_1^2 u - K_1(\alpha_1|u|^2 + \beta_1|v|^2)u , \\ jK_2 \partial_x v &= -\tfrac{1}{2}\nabla_1^2 v - K_2(\alpha_2|u|^2 + \beta_2|v|^2)v , \end{aligned} \tag{11}$$

where $\nabla_1^2 = \partial_y^2 + \partial_x^2$.

B. Equations for the pulse envelope

Now let the electromagnetic field be $\tilde{\mathbf{U}}(\mathbf{R},t) = \mathbf{U}(\mathbf{R},t)e^{-j\omega t}$, where $\mathbf{U}(\mathbf{R},t)$ is the six-vector of the slowly varying complex envelope. In order to predict the envelope's deformation during the propagation of a pulse through the chiral medium, we take the second-order approximation of the dispersion theory into account. We terminate the Taylor expansion of $\mathbf{U}(\mathbf{R},\tau)$ in the vicinity of $\tau = t$ to exclude terms higher than of the third order. Substituting these expansions into (3), we obtain

$$\hat{\Lambda}\tilde{\mathbf{U}} = \left[\Lambda \mathbf{U} + j\frac{\partial \Lambda}{\partial \omega}\partial_t \mathbf{U} - \frac{1}{2}\frac{\partial^2 \Lambda}{\partial \omega^2}\partial_t^2 \mathbf{U} \right] e^{-j\omega t} . \tag{12}$$

Let us now turn to the nonlinear term in Eq. (2). The terms involving $\partial_t \mathbf{U}$, $\partial_t^2 \mathbf{U}$, etc., can be neglected to simplify (5) because these small quantities, when multiplied by the components $\Delta\Lambda_{ij}$, are of a lower order of magnitude than the other terms. Thus, we approximately obtain

$$\begin{aligned} \nabla \times \mathbf{U} = &-j\omega\Lambda \mathbf{U} + \frac{\partial(\omega\Lambda)}{\partial\omega}\partial_t \mathbf{U} \\ &+ \frac{j}{2}\frac{\partial^2(\omega\Lambda)}{\partial\omega^2}\partial_t^2 \mathbf{U} - j\omega \mathbf{V}^{\mathrm{NL}} . \end{aligned} \tag{13}$$

We now introduce the Beltrami fields $\mathbf{Q}_{\pm}$ through the relations

$$\mathbf{Q}_{\pm} = \mathbf{E} \pm j\eta \mathbf{H} \mp \frac{\partial \eta_0}{\partial \omega}\partial_t \mathbf{H} , \tag{14}$$

where $\eta_0 = \sqrt{\mu/\epsilon}$. We emphasize that if μ and ϵ have the same frequency dependences, then $\partial\eta_0/\partial\omega = 0$ [37] and the definitions of $\mathbf{Q}_{\pm}$ used here reduce to those given in [5].

Correct to the terms containing ∂_t^3, $\Delta\Lambda_{ij}\partial_t H_j$, and $\Delta\Lambda_{ij}\partial_t^2 H_j$, (13) can be transformed to

$$\begin{aligned} \nabla \times \mathbf{Q}_{\pm} = \pm K_{1,2}\mathbf{Q}_{\pm} \pm j\frac{dK_{1,2}}{d\omega}\partial_t \mathbf{Q}_{\pm} \mp \frac{1}{2}\frac{d^2K_{1,2}}{d\omega^2}\partial_t^2 \mathbf{Q}_{\pm} \\ \pm f_{1,2}\mathbf{Q}_{\pm} , \end{aligned} \tag{15}$$

where $f_{1,2}$ are given by (9). It follows from (15) that, as before, only the nonlinear terms couple the Beltrami fields $\mathbf{Q}_{\pm}$.

Let us consider a one-dimensional electromagnetic field depending only on x and t (i.e., $\partial_y = \partial_z = 0$), and seek a solution of (15) by using the above factorizations (10), where the substitutions $u_l(\mathbf{r},x) = v_l(\mathbf{r},x) = 0$, $u(\mathbf{r},x) \to u(x,t)$, and $v(\mathbf{r},x) \to v(x,t)$ must be done. Operating next by analogy with the previous section, we obtain

$$\begin{aligned} jg_1(\partial_x + V_1^{-1}\partial_t)u &= -\tfrac{1}{2}\partial_t^2 u - g_1(\alpha_1|u|^2 + \beta_1|v|^2)u , \\ jg_2(\partial_x + V_2^{-1}\partial_t)v &= -\tfrac{1}{2}\partial_t^2 v - g_2(\alpha_2|u|^2 + \beta_2|v|^2)v , \end{aligned} \tag{16}$$

where $V_{1,2} = (dK_{1,2}/d\omega)^{-1}$ are group velocities and $g_{1,2} = (d^2K_{1,2}/d\omega^2)^{-1}$. In order to analyze (16) we introduce two new independent variables: $\xi = -x$ and $s = x/V_1 - t$. Some algebraic manipulations then reduce (16) to

$$\begin{aligned} jg_1\partial_\xi u &= -\tfrac{1}{2}\partial_s^2 u - g_1(\alpha_1|u|^2 + \beta_1|v|^2)u , \\ jg_2(\partial_\xi + a\partial_s)v &= -\tfrac{1}{2}\partial_s^2 v - g_2(\alpha_2|u|^2 + \beta_2|v|^2)v , \end{aligned} \tag{17}$$

where $a = V_1^{-1} - V_2^{-1}$.

C. Duality of the Beltrami representation

In order to obtain $\mathbf{E}$ and $\mathbf{H}$ from the Beltrami components $\mathbf{Q}_{\pm}$ we can make use of (14):

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{Q}_+ + \mathbf{Q}_-) , \tag{18}$$

$$\mathbf{H} + \frac{j}{\eta_0}\frac{\partial\eta_0}{\partial\omega}\partial_t \mathbf{H} = \frac{\mathbf{Q}_- - \mathbf{Q}_+}{2j\eta} . \tag{19}$$

Equation (18) is straightforward and requires no further comment. A simple way to solve the differential equation (19) is to apply an iteration procedure, assuming the dispersion correction $\partial\eta_0/\partial\omega$ to be sufficiently small. Then the right-hand side of (19) can be substituted into the left-hand side as a first approximation for $\mathbf{H}$ to get

$$\mathbf{H} \approx \frac{\mathbf{Q}_- - \mathbf{Q}_+}{2j\eta} - \frac{j}{2\eta_0^2}\frac{\partial\eta_0}{\partial\omega}(\partial_t \mathbf{Q}_+ - \partial_t \mathbf{Q}_-) . \tag{20}$$

Note that our Beltrami representation is not unique. Indeed, we can alternatively define

$$\mathbf{Q}_{\pm} = \mathbf{E} \pm j\eta \mathbf{H} - \frac{j}{\eta_0}\frac{\partial\eta_0}{\partial\omega}\partial_t \mathbf{E} ; \tag{21}$$

then the roles of $\mathbf{H}$ and $\mathbf{E}$ are interchanged. With the same initial and boundary conditions for $\mathbf{Q}_{\pm}$, (14) and (21) conform to physically different solutions corresponding to different initial conditions for the fields $\mathbf{E}$ and $\mathbf{H}$. The nonuniqueness of the Beltrami representation is analogous to the ambiguity inherent in the various electromagnetic potentials of classical electromagnetism [38].

III. STABILITY OF THE NONLINEAR PLANAR WAVE

Equations (11) have a partial solution in the form of a nonlinear planar wave as

$$u = u_0 \exp\{j(\alpha_1 u_0^2 + \beta_1 v_0^2)x + j\phi_u\} ,$$
$$v = v_0 \exp\{j(\alpha_2 u_0^2 + \beta_2 v_0^2)x + j\phi_v\} , \tag{22}$$

where u_0, v_0, ϕ_u, ϕ_v are specified real constants. Let us investigate the stability of (22) by applying an approach conventionally used for self-focusing in ordinary nonlinear dielectric media [39]. Consider the spatial evolution of a perturbation propagating along the initial wave vector to generalize the technique developed in [39] for a single nonlinear Schrödinger equation for our NCSE.

The perturbative solution of (11) can be obtained by replacing $u_0 \to u_0 + \delta u$ and $v_0 \to v_0 + \delta v$ in (11), where δu and δv are the small spatial variations. Equations for δu and δv are then found by neglecting all terms nonlinear with respect to δu and δv:

$$jK_1 \partial_x \delta u = -\tfrac{1}{2}\nabla_\perp^2 \delta u - \rho_1(\delta u + \delta u^*) - \rho_2(\delta v + \delta v^*) ,$$
$$jK_2 \partial_x \delta_v = -\tfrac{1}{2}\nabla_\perp^2 \delta v - \rho_3(\delta v + \delta v^*) - \rho_4(\delta u + \delta u^*) , \tag{23}$$

where $\rho_1 = K_1\alpha_1 u_0^2$, $\rho_2 = K_1\beta_1 u_0 v_0$, $\rho_3 = K_2\beta_2 v_0^2$, and $\rho_4 = K_2\alpha_2 u_0 v_0$.

Let us seek a solution of this set of equations in the form

$$\delta u = A \exp\{j(\mathbf{q}\cdot\mathbf{r} + \Gamma x)\} + B^* \exp\{-j(\mathbf{q}\cdot\mathbf{r} + \Gamma x)\} ,$$
$$\delta v = C \exp\{j(\mathbf{q}\cdot\mathbf{r} + \Gamma x)\} + D^* \exp\{-j(\mathbf{q}\cdot\mathbf{r} + \Gamma x)\} , \tag{24}$$

where $\mathbf{q}$ is a given vector in the YZ plane; Γ is the perturbation wave number to be sought; and A, B, C, D are the unknown constants.

Substitution of (24) into (23) leads to the matrix equation

$$\mathbf{M}\cdot\mathcal{A} = 0 ,$$

where

$$\mathbf{M} = \begin{bmatrix} \frac{q^2}{2} + K_1\Gamma - \rho_1 & -\rho_1 & -\rho_2 & -\rho_2 \\ -\rho_1 & \frac{q^2}{2} + K_1\Gamma - \rho_1 & -\rho_2 & -\rho_2 \\ -\rho_4 & -\rho_4 & \frac{q^2}{2} + K_2\Gamma - \rho_3 & -\rho_3 \\ -\rho_4 & -\rho_4 & -\rho_3 & \frac{q^2}{2} + K_2\Gamma - \rho_3 \end{bmatrix} \tag{25}$$

and $\mathcal{A}$ is the column vector consisting of the coefficients A, B, C, D. Simple manipulations reduce the characteristic equation for Γ to the biquadratic form as follows:

$$K_1^2 K_2^2 \Gamma^4 - q^2(K_2^2 X_1 + K_1^2 X_3)\Gamma^2 + q^4(X_1 X_3 - \rho_2\rho_4) = 0 , \tag{26}$$

where $X_i = q^2/4 - \rho_i$, $i = 1,3$.

If all scalar nonlinear coefficients are of the same sign, instability appears when one of the conditions

$$X_1 X_3 < \rho_2\rho_4 , \tag{27}$$

$$K_2^2 X_1 + K_1^2 X_3 < 0 \tag{28}$$

holds true. The inequalities (27) and (28) generalize the classical Bespalov-Talanov criteria [39] originally derived for media with the conventional Kerr nonlinearity. The conditions $\alpha_{1,2} > 0$ and $\beta_{1,2} > 0$ provide the fulfillment of one of the inequalities and, thus, are responsible for the occurrence of instabilities. As a result, the corresponding chiral medium is self-focusing.

Suppose $\alpha_{1,2} < 0$ and $\beta_{1,2} < 0$ in a certain medium. Then the solution (22) is stable under the additional restriction $\alpha_1\beta_2 \ge \alpha_2\beta_1$. In particular, such a situation occurs when neglecting the induced circular birefringence (i.e., $\alpha_1 = \alpha_2 = \beta_1 = \beta_2$) [21]. On the other hand, the condition $\alpha_1\beta_2 < \alpha_2\beta_1$ fulfills (26) and creates instabilities when

$$\frac{q^4}{16} - \frac{q^2}{4}(K_1\alpha_1 u_0^2 + K_2\alpha_2 v_0^2) < K_1 K_2 u_0 v_0 (\alpha_2\beta_1 - \alpha_1\beta_2) . \tag{29}$$

The instability of a wave beam in a defocusing chiral medium defined by (29) is analogous to the modulation instability of a wave packet investigated in [26]. However, it should be noted that the analysis given in [26] refers only to a special case. This special case is denoted by $u_0 = v_0$, $K_1 = K_2$, $\alpha_2 = \beta_1 = \kappa(1+\sigma)$, and $\alpha_1 = \beta_2 = \kappa(1-\sigma)$ in our notation, σ being a real quantity.

IV. CONSERVATION PRINCIPLES

Equations (17) and (11) can be reduced to the canonical Hamiltonian form [40], which makes the application of many standard techniques possible. Let us commence by considering (11). These equations are equivalent to

$$j\partial_x u = \alpha_2^{-1}\frac{\delta\mathcal{H}}{\delta u^*}, \quad j\partial_x v = \beta_1^{-1}\frac{\delta\mathcal{H}}{\delta v^*} , \tag{30}$$

where $\delta/\delta u^*$, etc., are variational derivatives and

$$\mathcal{H}(u,v;u^*,v^*)=\frac{1}{2}\int_{s_\perp}[\alpha_2K_1^{-1}|\nabla_\perp u|^2+\beta_1K_2^{-1}|\nabla_\perp v|^2 -\alpha_1\alpha_2|u|^4-2\beta_1\alpha_2|u|^2|v|^2 -\beta_1\beta_2|v|^4]ds_\perp\ . \quad (31)$$

The integral on the right-hand side of (31) has to be evaluated in the YZ plane. It can easily be shown that (30) takes the canonical Hamiltonian form with the Hamiltonian defined by (31) if we introduce the complex canonical variables $\tilde{u}=\sqrt{\alpha_2}u$ and $\tilde{v}=\sqrt{\beta_1}v$. We shall not, however, use these variables because they differ from u and v only by constant factors.

It follows from the existence of the canonical representation that equations of motion can be introduced and then determined as

$$\int_{s_\perp}|u|^2ds_\perp=\text{const},\quad \int_{s_\perp}|v|^2ds_\perp=\text{const},\quad \mathcal{H}=\text{const}\ , \quad (32)$$

assuming rapidly decreasing boundary conditions at infinity. The condition $\mathcal{H}=\text{const}$ gives us a sufficient criterion for self-focusing beams in chiral media. This criterion is the counterpart of the self-focusing criterion for ordinary dielectrics with Kerr nonlinearity [41–43].

The self-focusing effect should be observable if the inequality $\mathcal{H}<0$ holds on the boundary $x=x_0$. Specifying at $x=x_0$ the profile of the beam (for instance, Gaussian) and assuming the paraxial approximation [21,42] to be valid, we can obtain an equation for the variation of the propagated beam width from the condition $\mathcal{H}=\text{const}$.

The Lagrangian of a beam in a nonlinear chiral medium may be expressed through the Hamiltonian as

$$\mathcal{L}=\frac{1}{2j}\int_{s_\perp}[\beta_1 v\partial_x v^*-\beta_1 v^*\partial_x v+\alpha_2 u\partial_x u^* -\alpha_2 u^*\partial_x u]ds_\perp-\mathcal{H}(u,v;u^*,v^*)\ . \quad (33)$$

Knowing $\mathcal{L}$ allows us to formulate the variation principle for (11).

By analogy with (31) we can write the Hamiltonian of a wave packet described by (17) as

$$\mathcal{H}(u,v;u^*,v^*)=\frac{1}{2}\int_{-\infty}^{\infty}[\alpha_2 g_1^{-1}|\partial_s u|^2+\beta_1 g_2^{-1}|\partial_s v|^2 +ja\beta_1 v^*\partial_s v+ja\beta_1 v\partial_s v^* +\alpha_1\alpha_2|u|^4+2\beta_1\alpha_2|u|^2|v|^2 +\beta_1\beta_2|v|^4]ds\ . \quad (34)$$

Now the integration must be carried out along the real axis ($-\infty\le s\le\infty$), and the Lagrangian of a wave packet has the same form as that of a beam (33). For the special case of $\alpha_2=\beta_1$, the Hamiltonians (31) and (34) are available elsewhere [44].

The Hamiltonians (31) and (34) correspond to rapidly decreasing boundary conditions at infinity: the functions u and v are assumed to belong to the Schwartz space [45]. In other words, u and v are infinitely differentiable and decrease, along with their derivatives, faster than $|\mathbf{r}|^{-1}$ as $|\mathbf{r}|\to\infty$. The functionals (31) and (33) are permissible [45], and their variational derivatives are elements of the Schwartz space as well.

The case of boundary conditions of a finite density [45] is also of interest, for example, to analyze the interaction of bright and dark solitons (see Sec. V). Then we have to consider (11) with the boundary conditions $u\to u_0\exp(j\Phi)$, $u_0=\text{const}$, and $v\to 0$ as $|\mathbf{r}|\to\infty$. The functions $u-u_0\exp(j\Phi)$ and v are assumed to be elements of the Schwartz space. The Hamiltonian (30) must be modified for this case to

$$\mathcal{H}(u,v;u^*,v^*)=\frac{1}{2}\int_{s_\perp}[\alpha_2K_1^{-1}|\nabla_\perp u|^2+\beta_1K_2^{-1}|\nabla_\perp v|^2 -\alpha_1\alpha_2(|u|^2-u_0^2)^2 -2\beta_1\alpha_2|u|^2|v|^2 -\beta_1\beta_2|v|^4]ds_\perp\ . \quad (35)$$

The Hamiltonian (34) corresponding to set (17) must be modified similarly.

V. BELTRAMI-MAXWELL SOLITONS

Let us now proceed to consider soliton solutions of the nonlinear wave equations (17) and (11). Suppose the chiral medium has the simple Kerr nonlinearity determined by the relations $\Delta\epsilon=\lambda|E|^2$, $\Delta\mu=\Delta\gamma=0$, with λ as a known quantity. Assuming the effect of induced circular birefringence to be negligible [21], we can set $\alpha_1=\alpha_2=\beta_1=\beta_2=\kappa$. Let us also restrict ourselves to the two-dimensional case $\partial_z=0$.

We will also assume that the polarization state of the electromagnetic field is almost circular, i.e., one of the two Beltrami fields is predominant. Let $|v|^2\ll|u|^2$, so we can neglect the influence of the v component of the field on its u component. As a result, the governing differential equation for the u component is a single nonlinear Schrödinger equation that can be exactly solved using the inverse scattering technique [43]. The governing differential equation for the v component similarly transforms into a single nonlinear Schrödinger equation, but with a spatially inhomogeneous internal potential $|u|^2$ acting as the source term. A possible approximate approach is available in [46].

Our objective is to apply two other approaches. The first is based on the general Karpman-Maslov perturbation method [28]. In this method, the v component is considered as a perturbation of the u soliton, which results in the linearization of the equation for the v component (Secs. V A and V B). The second approach uses the Hamiltonian formalism and paraxial approximation for a Gaussian beam (Sec. V C).

A. Bright solitons in beams

Let us analyze some features of the substantially nonlinear phase of beam self-focusing, when the effect of dispersion spreading compensates for nonlinear focusing and results in the appearance of solitons. In accordance with the Karpman-Maslov perturbation method [28], it is possible to neglect the term $|v|^2u$ in the first equation of (11) and the term $|v|^2v$ in the second one. Then the system (11) simplifies to

$$jK_1\partial_x u = -\tfrac{1}{2}\partial_y^2 u - K_1\kappa|u|^2 u \ , \tag{36}$$

$$jK_2\partial_x v = -\tfrac{1}{2}\partial_y^2 v - K_2\kappa|u|^2 v \ , \tag{37}$$

with the u in (37) being a solution of (36). Such a procedure has been used to examine the branching of vector solitons in fiber waveguides [16].

Let us assume $\kappa > 0$ and take a single-soliton solution of (36) in the form [47]

$$u = \frac{u_0}{\cosh[u_0\sqrt{\kappa K_1}(y+2\nu x)]} \times \exp\left\{ j\phi_0 + j\frac{\kappa u_0^2}{2}x - 2jK_1\nu(y+\nu x) \right\} , \tag{38}$$

with u_0, ϕ_0, and ν as constants. This soliton acts as a waveguide for the v of (37), whose discrete eigenfunction spectrum has been given in [48] as

$$v_n(x,y) = \exp\{-jE_n x - 2jK_2\nu(y+\nu x)\}(1-\zeta^2)^{(p-n)/2} \times F\left[-n, 2p-n+1, p-n+1, \frac{1-\zeta}{2} \right] . \tag{39}$$

Here, $\zeta = \tanh(u_0\sqrt{\kappa K_1}(y+2\nu x))$ and $F[\dots]$ is the hypergeometric function,

$$p = \frac{1}{2}\left[\left[1 + \frac{8K_2}{K_1} \right]^{1/2} - 1 \right], \quad E_n = -\frac{u_0^2\kappa K_1}{2K_2}(n-p)^2 , \quad n = 0, \pm 1, \pm 2, \dots . \tag{40}$$

In view of the assumption $|v|^2 \ll |u|^2$ taken above, (38)–(40) describe solitons in a chiral medium, when the initial polarization state of the launched beam is close to circular.

The number of eigenfunctions in the discrete spectrum is determined by the inequality $n < p$. There is a single eigenfunction (39) with $n = 0$ for $K_2 < K_1$. In this case, the soliton's wave numbers for its u and v components turn out to be different and, thus, circular birefringence is displayed by the soliton. For $K_1 < K_2 < 3K_1$ two eigenfunctions exist (with $n = 0$ and $n = 1$), resulting in trirefringent solitons. Quadrirefringence should be observable (eigenfunctions with $n = 0, 1, 2$) when $3K_1 < K_2 < 6K_1$, and so on.

The multirefringence of solitons is conditioned by the joint action of nonlinearity and chirality. A similar effect occurs for Zeeman solitons in quasiresonant nonlinear magneto-optics [49]. The physical mechanism of the multirefringence of solitons is the following: the higher eigenfunctions (39) are coupled soliton states, each being excited in a conventional nonlinear medium above its own energy threshold. As follows from (40), in a chiral medium these energy thresholds are replaced by thresholds with respect to the degree of chirality (as quantified by the ratio K_2/K_1). Consequently, at a given energy and sufficiently large chiral parameter $\gamma(\omega)$, the phenomenon of multirefringence of solitons occurs.

The chiral admittance of naturally occurring chiral media is small [34], even though its effects are easily observable and have been technologically significant since the mid-1800s [1]. This allows us to approximate p in (40) as $p = 1 + \theta$, with $\theta = 4\gamma/3\sqrt{\epsilon\mu}$. Typically, θ lies between 10^{-5} and 10^{-4} [50]. Thus, only one soliton exists for the one quasicircular polarization and two solitons exist for the other one, assuming that $\gamma > 0$. Of particular interest is the evaluation of the critical power density W_{cr} of the soliton at $n = 1$ in (39); for small θ,

$$v_1 \sim C\tanh(u_0\sqrt{\kappa K_1}y)\cosh^{-\theta}(u_0\sqrt{\kappa K_1}y) , \tag{41}$$

where C is a constant. Correct to first order in θ,

$$W_{cr} \cong W_u \left[1 + \left[\frac{C}{u_0} \right]^2 \theta^{-1} \right] , \tag{42}$$

where $W_u = cu_0/8\pi\sqrt{\kappa K_1} = c\Lambda_u^2/4\pi^3\lambda\Delta_u$ is the power density of the u soliton at $v = 0$, Λ_u is the wavelength, c is the speed of light in free space, and Δ_u is the soliton's width. Note that W_{cr} in (42) tends to infinity in the achiral limit (i.e., as $\theta \to 0$) and consequently, the excitation of the soliton being considered proves to be impossible when chirality is absent. Taking $\Lambda_u = 500$ nm and $\Delta_u = 100$ μm, we see that W_u lies between 10^6 and 10^8 W/cm when the nonlinear index coefficient λ varies between 10^{-11} and 10^{-13} esu [21]. Then, in view of $(C/u_0)^2 \sim 10^{-4}$, we find that the value of W_{cr} turns out to be comparable with W_u, even at small θ; which is to say that the soliton under consideration is observable. Thus, even small chirality leads to a qualitatively novel effect: an asymmetry of the solitonic spectrum with respect to the handedness of the field.

B. Bright solitons in wave packets

Let us now consider the nonlinear phase of pulse propagation resulting in the occurrence of space-time solitons. For this purpose we return to (17).

Let us assume that $g_1\kappa > 0$ in the nonlinear Schrödinger equation for the u component and take the single-soliton solution of this equation in the form analogous to (38):

$$u = \frac{u_0}{\cosh(u_0 s\sqrt{|g_1|})}\exp\left\{ j\phi_0 - j\frac{\kappa u_0^2}{2}\xi \right\} . \tag{43}$$

Then we can write the discrete eigenfunction spectrum for the v component as

$$v_n(x,y)=\exp\left\{jg_2as-j(E_n+\tfrac{1}{2}g_2a^2)\xi\right\}(1-\zeta^2)^{(p-n)/2}\times F\left[-n,2p-n+1,p-n+1,\frac{1-\zeta}{2}\right], \quad (44)$$

with $\zeta=\tanh(u_0s\sqrt{|\kappa g_1|})$. E_n and p can be obtained from (40) after the replacements $K_{1,2}\to g_{1,2}$ and $\kappa\to-\kappa$ have been carried out.

In a chiral medium it is possible that the coefficients g_1 and g_2 are of opposite signs. For instance, at low frequencies, where $\epsilon(\omega)$ and $\mu(\omega)$ are almost constant while $\gamma(\omega)\sim\omega$, the dispersive properties of the chiral medium are determined chiefly by the frequency dependence of $\gamma(\omega)$. Let $g_1\kappa<0$ and $g_2\kappa>0$. Then, eigenfunctions of the kind given on the left-hand side of (44) are absent; thus, the soliton part of the field contains $\mathbf{Q}_+$ but not $\mathbf{Q}_-$, and the helicity of the soliton part does not depend on the initial conditions. The other Beltrami field $\mathbf{Q}_-$ can be present in the nonsoliton background and/or forms a dark soliton, depending on the initial conditions.

Also of special interest may be the case when the frequency dependence of γ can be ignored and consequently, $g_1\to g_2$. With this condition and the above-established restrictions on α_i and β_i, the system (17) can be reduced, by analogy with [14], to the canonical Manakov system [22].

C. Interaction of bright and dark solitons

The interaction of bright and dark solitons in nonlinear optical fibers was numerically studied by others [51,27], who showed that a dark soliton in a defocusing medium exhibits the property of self-focusing. We discuss here the manifestation of this effect in chiral media, and —in contrast to [51,27]—we develop an analytical theory based on the Hamiltonian formalism and paraxial approximation.

Let us consider a beam launched into a defocusing chiral medium with $\kappa<0$. Equation (36) is amenable to exact integration and yields the dark soliton [52]

$$u=u_0\tanh[u_0\sqrt{|\kappa|K_1}y]\exp\{j\kappa u_0^2x\}\ . \quad (45)$$

Consistently with the framework of paraxial approximation, the function v can be represented by a Gaussian beam as [53]

$$v=\frac{v_0}{\sqrt{f(x)}}\exp\left\{-\frac{y^2}{a_0^2f^2}-jK_2\frac{y^2}{2f}\frac{df}{dx}\right\}, \quad (46)$$

where a_0 is the known beam waist and $f(x)$ is the function being sought. Substituting (45) and (46) into the Hamiltonian (34), and using the conservation law $\mathcal{H}=$const, we obtain

$$\left[\frac{df}{dx}\right]^2+\frac{1}{R_D^2f^2}+\frac{1}{R_{\rm NL}^2f}-\frac{F(f)}{R_{\rm NL}^2}=\text{const}\ . \quad (47)$$

In this equation,

$$R_D=\tfrac{1}{2}K_2a_0^2,\quad R_{\rm NL}=\frac{a_0}{v_0}\left[\frac{K_2}{2\sqrt{2}|\kappa|}\right]^{1/2}$$

are respectively the diffraction and the nonlinear radii, while

$$F(f)=4\left[\frac{2}{\pi}\right]^{1/2}\left[\frac{u_0}{v_0}\right]^2\int_0^\infty\cosh^{-2}(Bf\vartheta)e^{-\vartheta^2}d\vartheta \quad (48)$$

and $B=u_0a_0\sqrt{|\kappa|K_2/2}$. The integral in (48) cannot be evaluated analytically. Its general properties and different representations are given in Appendix B.

Taking the wave front in the initial section to be planar, we should supplement (47) with boundary conditions $f|_{x=0}=1$ and $(df/dx)|_{x=0}=0$. This leads to

$$\left[\frac{df}{dx}\right]^2=\Pi(f)\ , \quad (49)$$

where

$$\Pi(f)=\frac{f^2-1}{f^2R_D^2}+\frac{f-1}{fR_{\rm NL}^2}+\frac{F(f)-F(1)}{R_{\rm NL}^2} \quad (50)$$

with the third term on the right-hand side accounting for the role of the dark soliton.

Qualitative analysis of (49) can be performed by analogy with [54]. The behavior of the right-hand side of (46) is fully determined by the real positive roots of the function $\Pi(f)$. When the third term is absent from the right-hand side of (50), the only root is $f=1$. In this case the function $\Pi(f)$ is negative if $f<1$; otherwise it is positive and $\Pi(f)\to$const as $f\to\infty$. Thus, starting with the initial value $f=1$, f will increase indefinitely, resulting in self-defocusing of the wave. The incorporation of the dark soliton implies that the equation $\Pi(f)=0$ may have an additional root. When $f=1$ is a multiple root, waveguiding will occur [54].

Let us analyze the equation $\Pi(f)=0$ in the vicinity of the waveguiding regime where the linear approximation of the function $F(f)$ can be used: $F(f)\approx F(1)+D(1-f)$, $D=-(dF/df)|_{f=1}$. Then the right-hand side of (50) becomes quadratic in f. One of the two roots is $f=1$, the other is $f=f_0$, where

$$f_0=\frac{R_{\rm NL}^2}{2D}(b+\sqrt{b^2+4DR_D^{-2}R_{\rm NL}^{-2}})\ , \quad (51)$$

and $b=R_D^{-2}+R_{\rm NL}^{-2}$, $D>0$ in accordance with Appendix B. The waveguiding condition can be found by enforcing the equality $f_0=1$. This is imposed on the amplitude of the dark soliton u_0 to delineate the waveguiding regime through

$$D=4\left[\frac{2}{\pi}\right]^{1/2}\left[\frac{u_0}{v_0}\right]^2B\int_0^\infty\cosh^{-3}(Bf\vartheta)\vartheta e^{-\vartheta^2}d\vartheta=1+\frac{2R_{\rm NL}^2}{R_D^2}\ . \quad (52)$$

It follows from (51) that $f_0>1$ at $D<1+2R_{\rm NL}^2/R_D^2$. Hence, the function $\Pi(f)$ is positive over the interval

$1<f<f_0$. It implies that the beam (46) will not spread indefinitely but only up to the value $f=f_0$, after which it will turn back to the initial state, etc. Thus, the waveguide will be oscillatory. The situation will be different if $D>1+2R_{\mathrm{NL}}^2/R_D^2$, whence $f_0<1$ and $\Pi(f)>0$, $f_0<f<1$, and the beam will be periodically compressed from the initial point $f=1$ up to the point $f=f_0$.

Thus, the dark soliton of a given polarization state changes the focusing characteristics of a chiral medium. It may be responsible for creating a waveguiding regime, as well as for the focusing of beams with the other polarization state. The features revealed above manifest themselves when the dark soliton's amplitude exceeds some crucial value. If the amplitude is less than the critical value, all roots of $\Pi(f)$ except $f=1$ vanish and consequently, the medium remains defocusing for both polarization states [55]. We remark here that equivalent results can be similarly obtained for a wave packet with the Gaussian envelope and three-dimensional ring dark solitons, whose existence has been predicted by Kivshar and Yang [56].

VI. DISCUSSION

In this paper we have developed a theory of the self-action of waves in nonlinear chiral media. The use of Beltrami fields in nonlinear problems turned out to be an effective instrument of analysis, allowing the reduction of the initial problem to a compact system of nonlinear coupled Schrödinger equations. Our approach is based on a variant of the general Karpman-Maslov perturbation method [28]. This approach is characterized by the use of a nonlinear Schrödinger equation to perturb another nonlinear Schrödinger equation.

Another variant of the Karpman-Maslov method based on the exact Manakov solution of the vector nonlinear Schrödinger equation [22] may also be of interest. In this approach, the unperturbed equation is obtained from (11) after assuming that $K_1=K_2$ and $\alpha_1=\alpha_2=\beta_1=\beta_2=\kappa$. This approach is applicable only for a weakly chiral medium, but it allows the relationship between u and v to be arbitrary. We also point out that the Hirota method [20], as well as explicit numerical solution procedures, remain unexplored in the present context.

The joint action of chirality and nonlinearity results in several interesting physical effects, such as multirefringent solitons, the availability of solitons with specified circular polarization states, and the focusing of solitons of specific polarization states in a defocusing medium under the effect of the oppositely polarized dark soliton. Similar effects can be predicted for electromagnetic pulses as well: for instance, the compression of a left-circularly polarized pulse is possible simultaneously with the decompression of the right-circularly polarized pulse. Thus, the degree of chirality controls the polarization states of solitons.

The effects discussed in our paper can also manifest themselves in certain noncentrosymmetric optically active crystals (for instance, quartz and $LiIO_3$). The fact that the optical anisotropy has not been taken into account by us is not critical: all equations derived above are also valid for anisotropic crystals if the radiation is taken to propagate along a crystallographic axis. Of even greater interest for the Beltrami-Maxwell soliton observation are synthetic chiral media. Linear composite chiral materials are studied extensively (see [2,3] and references therein). As also mentioned above, a composite material consisting of a linear gain host medium and small chiral scatterers have been proposed [11] to reduce the distributed feedback laser threshold gain.

A synthetic nonlinear chiral medium can be created as a dilute suspension of chiral objects with constitutive parameters ϵ, μ_0, and γ in an achiral, nonmagnetic, nonlinear host medium characterized by the nonlinear permittivity $\epsilon_1+\lambda|E_{\mathrm{loc}}|^2$, where E_{loc} is the local electric field in the composite medium. It is convenient to quantitate the physical properties of such a medium using the Maxwell Garnett approach [2]. Generalization of this approach to the nonlinear case gives us an effective medium, the Maxwell Garnett estimates of whose constitutive parameters are $\epsilon_{\mathrm{eff}}+\lambda_{\mathrm{eff}}|E|^2$, μ_{eff}, and γ_{eff}. It should be emphasized that the distinction of E_{loc} from the macroscopic mean field E has to be taken into account in the nonlinear term: $E_{\mathrm{loc}}=LE$ with L as a constant coefficient. Assuming the chirality parameter to be small, we find that [3] $\epsilon_{\mathrm{eff}}=\epsilon_1+g(\epsilon-\epsilon_1)$, $\mu_{\mathrm{eff}}=\mu_0$, and $\gamma_{\mathrm{eff}}=g\gamma$, while

$$\lambda_{\mathrm{eff}}=\lambda\left[1-g+g\left[1-\frac{2+N}{3N}g\right]\frac{\epsilon-\epsilon_1}{\epsilon_1}\right]L^2\,, \tag{53}$$

where $g=3N\epsilon_1/[\epsilon+2\epsilon_1-N(\epsilon-\epsilon_1)]$, N is the volumetric proportion of chiral scatterers, and $L=g(\epsilon+2\epsilon_1)/3N\epsilon_1$. As N increases from 0, γ_{eff} increases from zero towards γ and λ_{eff} decreases from λ towards zero. Using this approach thus, one can choose a value of N which provides acceptable characteristics of a composite medium with respect to both chirality and nonlinearity.

Finally, there are some interesting physical phenomena characteristic of nonlinear chiral media but which were beyond the scope of this paper. An example is furnished by the dependence of group velocity on pulse intensity. In nonlinear dielectrics this dependence leads to the effect of the "turning-over" of pulses, as well as to the formation of shock waves of envelopes at their leading or trailing edges [57]. We conjecture that shock waves of different polarization states will have different group velocities in a chiral medium. The result would be the spatial separation of shock waves as they travel through the medium.

Novel effects may be expected in periodically inhomogeneous, nonlinear chiral media. Analysis of Bragg diffraction by a linear chiral grating shows certain features of the diffraction pattern attributable solely to the degree of chirality [58]. This causes us to anticipate that the joint action of the chirality, periodicity, and nonlinearity will significantly extend the range of technologically exploitable phenomena.

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APPENDIX A

Here we give the derivation of the system (11). After substituting the factorizations (10) into (8) we obtain

$$\begin{aligned}&\frac{1}{\sqrt{2}}(j\partial_y-\partial_z)u=(K_1+f_1)u_l\ ,\\&-\frac{1}{\sqrt{2}}j\partial_x u+\partial_z u_l=\frac{1}{\sqrt{2}}f_1 u\ ,\\&\frac{1}{\sqrt{2}}\partial_x u-\partial_y u_l=\frac{j}{\sqrt{2}}f_1 u\ ,\end{aligned}\tag{A1}$$

and three similar relations for v and v_l. Assuming u_l to be small, we can further neglect the nonlinear term f_1 in the first equation of (A1). In order to eliminate u_l now, let us differentiate the first one of (A1) with respect to z and use the result for $\partial_z u_l$ in the second equation of (A1). Next, we differentiate the first equation of (A1) with respect to y and use $\partial_y u_l$ from the third equation of (A1). These two manipulations yield

$$\nabla_\perp^2 u=2jK_1\partial_x u+2f_1K_1u\ ,\tag{A2}$$

which is identical to the first equation of the system (11). The equation for v is obtained in the same way. Once the system (11) has been solved, it is not difficult to find u_l and v_l from (A1) and the corresponding system for v and v_l.

Normally, equations for slow amplitudes in nonlinear optics are derived from a wave equation for the electric field (see, for instance, [14]). The longitudinal field components are neglected. This is possible only because the total electric field as well as its transverse component obey identical wave equations. A very different situation arises with the Beltrami fields, and therefore the derivation of (11) is significantly different.

APPENDIX B

The function $F(f)$ from Eq. (48) correct to a constant factor is given by the integral

$$I(f)=\int_0^\infty \cosh^{-2}(Bf\vartheta)e^{-\vartheta^2}d\vartheta\ ,\tag{B1}$$

at real positive values of f. Note that $I(f)$ is always positive and $dI/df<0$. The function $I(f)$ can be expanded into an asymptotic series in terms of powers of f^{-1}. To do this, we make use the power series expansion in $X=\exp(-2Bf\vartheta)$ of the function $\cosh^{-2}(Bf\vartheta)\equiv 4X(1+X)^{-2}$. Then we obtain

$$\begin{aligned}I(f)&=4\lim_{\theta\to 0}\sum_{m=0}^{\infty}(-1)^{m+1}m\int_0^\infty \exp\{-\vartheta^2-2mBf\vartheta\}d\vartheta\\&=2\sqrt{\pi}\lim_{\theta\to 0}\sum_{m=1}^{\infty}(-1)^{m+1}m\exp\{(Bmf)^2\}\\&\qquad\times\operatorname{erfc}(Bmf+\theta)\ ,\end{aligned}\tag{B2}$$

where erfc(x) is the additional probability integral [59]. Using the asymptotic expression for erfc(x) at large x, performing some algebraic manipulations, and taking the limit $\theta\to 0$, we finally find that

$$I(f)\simeq\frac{1}{Bf}\left[1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left[\frac{\pi}{2Bf}\right]^{2n}\times(2^{2n}-1)|B_{2n}|\right]\ ,\tag{B3}$$

where B_{2n} are the Bernoulli numbers. In the limit $f\to\infty$, it follows that $I(f)\to 0$ as f^{-1}.

For small f, the integral $I(f)$ may be developed as the Taylor series

$$I(f)=\sqrt{\pi}\sum_{n=0}^{\infty}\frac{2^{n+1}(2^{2n+2}-1)B_{2n+2}}{(2n+2)!!}(Bf)^{2n}\ .\tag{B4}$$

This expansion shows that $I(0)=\sqrt{\pi}/2$.

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